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A Gravity-Wave Problem with the Upstream Difference Method

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In numerous applications, upstream differencing has been used to treat the horizontal advection terms in primitive-equation models of the atmosphere. Though solutions to these equations are characterized by the presence of gravity waves, the velocity of these waves is not taken into account in formulating the difference equations. Determination of the "upstream" direction is based upon consideration of the transport velocity alone. With this approach, very small time steps are required to maintain computational stability when the gravity wave speed is large compared to the advecting speed. For the modeler who wishes to retain upstream differencing in such cases, a simple method is presented to enlarge the stability region.

1. INTRODUCTION

For many years the upstream differencing technique has provided an attractively simple alternative for approximating the advection terms in primitive-equation models of the atmosphere. Though the technique has been heavily criticized for its low-order error which acts as a dissipating agent [1], a number of research efforts in meteorology have used upstream differencing with reported success (e.g., [2–5]). Since the upstream scheme is most often used to treat the horizontal component of atmospheric flow and since the precise nature of the true horizontal diffusion process is not known, it may be that the inherent numerical diffusion provides some reasonable approximation to reality.

It is the authors' contention that the principal shortcoming of the upstream method lies not in its dissipative character but in its inability in many situations to deal effectively with gravity-wave-generating systems such as the primitive equations of meteorology. The problem arises from the fact that direct application of the upstream method takes account only of the advecting velocity and not of the gravity-wave velocity. Such an approach leads to severe restrictions on the size of the time step in order to maintain computational stability whenever the gravity-wave velocity is large in magnitude relative to the transport velocity. This fact may explain the current usage of various numerical techniques (e.g., splitting methods, semi-implicit methods) designed to maintain stability in meteorological models that employ upstream differencing.

A case study will be presented to illustrate the computational difficulties incurred in applying the upstream method to wave-type equations, and a simple procedure will

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be given to enlarge the stability region without seriously affecting the truncation error of the scheme.

2. Case Study

For our analysis we consider the linearized, one-dimensional shallow water wave equations

$$\frac{\partial u}{\partial t} = -U \frac{\partial u}{\partial x} - \frac{\partial \phi}{\partial x}, \tag{1a}$$

$$\frac{\partial \phi}{\partial t} = -U \frac{\partial \phi}{\partial x} - gH \frac{\partial u}{\partial x} \tag{1b}$$

for $t \geqslant 0$, $0 \leqslant x \leqslant \pi$, where u is the perturbation velocity, ϕ the geopotential, U the (constant) mean velocity, g the acceleration of gravity, and H the (constant) mean height. We shall assume periodicity in x and thereby avoid the problems associated with the treatment of lateral boundaries. Equations (1a) and (1b) are easily transformed into a pair of uncoupled equations by first multiplying (1b) by $(gH)^{-1/2}$ and then taking both the sum and the difference of the equations for $\partial u/\partial t$ and $\partial[(gH)^{-1/2}\phi]/\partial t$ to get

$$\frac{\partial \xi}{\partial t} = -(U + c) \frac{\partial \xi}{\partial x}, \tag{2a}$$

$$\frac{\partial \eta}{\partial t} = -(U - c) \frac{\partial \eta}{\partial x}, \tag{2b}$$

where

$$\xi = u + (gH)^{-1/2}\phi, \tag{3a}$$

$$\eta = u - (gH)^{-1/2}\phi, \tag{3b}$$

and

$$c = (gH)^{1/2}. \tag{4}$$

Equations (2a) and (2b) each take the form of a simple transport equation which can be solved by straightforward application of upstream differencing taking into account the direction of velocities $U + c$ and $U - c$. Clearly, $c > |U|$ implies that "upstream" in (2a) will be opposite in direction from that in (2b). The scheme will be stable provided the Courant–Friedrichs–Levy criterion is satisfied, i.e.,

$$\frac{(|U| + c)\,\Delta t}{\Delta x} \leqslant 1. \tag{5}$$

In treating primitive-equation models of meteorology, we are not afforded the luxury of such a convenient transformation and instead must deal with a coupled system of equations analogous to (1a) and (1b) but more complicated in various respects. In order to illustrate the problem in applying upstream differencing directly to the primitive equations, we shall apply the method to the simplified system (1) and analyze the resulting scheme for stability.

Let u_j^n denote the finite-difference approximation to $u(t, x) = u(n\Delta t, j\Delta x)$, where Δt and Δx $(=\pi/J)$ are the temporal and spatial grid increments, respectively, and define ϕ_j^n in a similar manner. Assume $U > 0$. Then a direct upstream-differencing approach, based on consideration of the transport velocity direction, yields the equations

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -U \frac{u_j^n - u_{j-1}^n}{\Delta x} - \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x}, \tag{6a}$$

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = -U \frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} - gH \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}, \tag{6b}$$

for $n = 0, 1, 2,\ldots$ and $j = 0, 1,\ldots, J$, where we have used centered differences for the "non-transport" terms. Instead of analyzing (6) directly, we can transform the discrete system in the same way that we transformed the continuous system. This approach leads to the uncoupled difference equations

$$\frac{\xi_j^{n+1} - \xi_j^n}{\Delta t} = -U \frac{\xi_j^n - \xi_{j-1}^n}{\Delta x} - c \frac{\xi_{j+1}^n - \xi_{j-1}^n}{2\Delta x}, \tag{7a}$$

$$\frac{\eta_j^{n+1} - \eta_j^n}{\Delta t} = -U \frac{\eta_j^n - \eta_{j-1}^n}{\Delta x} + c \frac{\eta_{j+1}^n - \eta_{j-1}^n}{2\Delta x}. \tag{7b}$$

Since ξ_j^n and η_j^n are just linear combinations of u_j^n and ϕ_j^n, the stability criteria for systems (6) and (7) will be equivalent. The advantage in dealing with (7) is that each equation may be analyzed separately.

To carry out the stability analysis, we first consider (7a) and use the Fourier series approach (e.g., [6]) assuming ξ_j^n to take the form

$$\xi_j^n = A^n e^{ikj\Delta x}. \tag{8}$$

Substitution of (8) in (7a) yields the characteristic equation

$$A - 1 = -\kappa_1 (1 - e^{-ik\Delta x}) - \kappa_2 i \sin k\Delta x, \tag{9}$$

where

$$\kappa_1 = U \frac{\Delta t}{\Delta x} \tag{10}$$

and

$$\kappa_2 = c\,\frac{\Delta t}{\Delta x}. \tag{11}$$

Scheme (7a) will be stable if and only if $|A| \leqslant 1$. A straightforward calculation shows that the equivalent criterion $|A|^2 \leqslant 1$ may be written as

$$-2\kappa_1(1-\kappa_1) + (2\kappa_1\kappa_2 + \kappa_2^2)\,(1+\cos k\Delta x) \leqslant 0. \tag{12}$$

Inequality (12) must hold for all wavenumbers k, with the most stringent condition occurring for $\cos k\Delta x = 1$ in which case the criterion for stability becomes

$$-2\kappa_1(1-\kappa_1) + 2(2\kappa_1\kappa_2 + \kappa_2^2) \leqslant 0 \tag{13}$$

or

$$(\kappa_1 + \kappa_2)^2 \leqslant \kappa_1. \tag{14}$$

In terms of the basic parameters, (14) is equivalent to the condition

$$\left(1+\frac{c}{U}\right)(U+c)\,\frac{\Delta t}{\Delta x} \leqslant 1. \tag{15}$$

This places a more severe restriction on the size of Δt than does (5), most significantly when c becomes large compared to U. For illustrative purposes we shall stay with the κ_1, κ_2 formulation and rewrite (14) as

$$0 \leqslant \kappa_2 \leqslant \kappa_1^{1/2} - \kappa_1. \tag{16}$$

(The upper and lower bounds imply that κ_1 must be $\leqslant 1$ in order for the possibility of a stable solution to exist.) The stability region in terms of κ_1 and κ_2 is shown by the shaded area in Fig. 1.

We now must determine under what conditions (7b) will be stable. For (7b), inequality (12) is replaced by

$$-2\kappa_1(1-\kappa_1) + (-2\kappa_1\kappa_2 + \kappa_2^2)(1+\cos k\Delta x) \leqslant 0. \tag{17}$$

We consider two cases: Case i: $(-2\kappa_1\kappa_2 + \kappa_2^2) \leqslant 0$ [or $\kappa_2 < 2\kappa_1$]. Here the most stringent condition in (17) occurs for two-grid-interval waves ($\cos k\Delta x = -1$) in which case the criterion for stability becomes

$$-2\kappa_1(1-\kappa_1) \leqslant 0 \tag{18}$$

or

$$\kappa_1 = U\,\frac{\Delta t}{\Delta x} \leqslant 1. \tag{19}$$

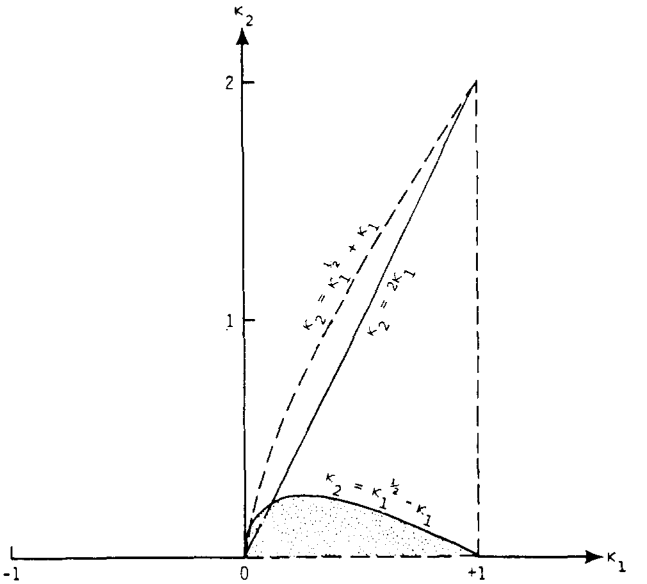

FIG. 1. The stability region for (7a) (shaded area) and for (7b) (outlined by the dashed line) when $U > 0$. The intersection of the two regions (shaded area) is the stability region for the combined system (7).

Case ii: $(-2\kappa_1\kappa_2 + \kappa_2^2) > 0$ [or $\kappa_2 > 2\kappa_1$]. In this case the longest waves ($\cos k\Delta x = 1$) determine the criterion

$$-2\kappa_1(1-\kappa_1) + 2(-2\kappa_1\kappa_2 + \kappa_2^2) \leqslant 0 \tag{20}$$

or

$$(\kappa_1 - \kappa_2)^2 \leqslant \kappa_1. \tag{21}$$

From (21) it follows that

$$0 \leqslant \kappa_2 \leqslant \kappa_1 + \kappa_1^{1/2}. \tag{22}$$

The stability region for (7b), as determined by (19) and (22), is bounded by the dashed line in Fig. 1. This line is seen to encompass the shaded area which thus represents the stability region for the system (7). Hence, (7), or equivalently (6), is stable if and only if (15) is satisfied.

A corresponding analysis for $U < 0$ ($\kappa_1 < 0$) can be carried out to determine a stability region symmetric to the shaded area of Fig. 1 with respect to the κ_2-axis.

The results of the above stability analysis firmly establish the need to enlarge the stability region, the need being greatest for those cases in which $c \gg |U|$. In the following section a procedure is presented for modifying the finite-difference scheme

(6) in such a way as to institute (5) as the stability criterion and thereby accomplish our goal of easing the restriction on Δt.

3. A Remedial Measure

Our approach is to deal with the stability problem through modification of the (u, ϕ) equations rather than to directly form stable finite-difference analogs of (2) since in most practical applications, a convenient (ξ, η) transformation is not available to us.

By use of Taylor series expansions, it is easily shown that the truncation error in (6a) takes the form

$$T = \Delta x \left\{ \frac{1}{2} \left[U(1-\kappa_1) - c\kappa_2 \right] \frac{\partial^2 u}{\partial x^2} - \kappa_1 \frac{\partial^2 \phi}{\partial x^2} \right\}_j^n + O[(\Delta t)^2] + O[(\Delta x)^2]. \quad (23)$$

(The order of the error is as low as it can be and still provide a consistent finite-difference approximation to (1).) The order of accuracy, then, will not be lowered if we append to the right-hand side of (6a) a diffusion-type term of the form

$$K \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} = K \left\{ \left(\frac{\partial^2 u}{\partial x^2} \right)_j^n + O[(\Delta x)^2] \right\} \quad (24)$$

provided that K is $O(\Delta x)$. Suppose such a term is added to (6a) with an analogous approximation to $K\, \partial^2\phi/\partial x^2$ added to (6b). If the diffusion is arbitrarily assigned the form

$$K = \tfrac{1}{2} c \Delta x, \quad (25)$$

the modified difference equations corresponding to (7) become

$$\frac{\xi_j^{n+1} - \xi_j^n}{\Delta t} = -U \frac{\xi_j^n - \xi_{j-1}^n}{\Delta x} - c \frac{\xi_j^n - \xi_{j-1}^n}{\Delta x}, \quad (26a)$$

$$\frac{\eta_j^{n+1} - \eta_j^n}{\Delta t} = -U \frac{\eta_j^n - \eta_{j-1}^n}{\Delta x} + c \frac{\eta_{j+1}^n - \eta_j^n}{\Delta x}. \quad (26b)$$

Equation (26a) is the usual upstream difference approximation to a transport equation with advecting velocity $U + c$ for which the stability criterion is well known to be

$$(U + c) \frac{\Delta t}{\Delta x} \leqslant 1. \quad (27)$$

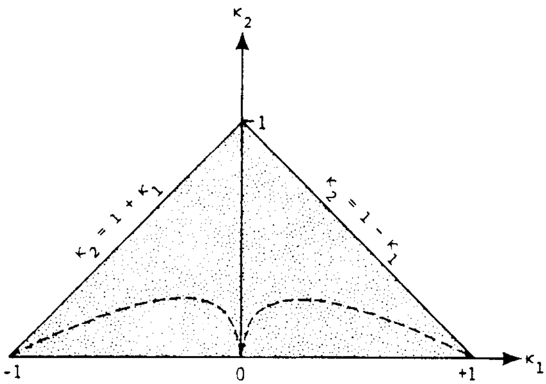

FIG. 2. The stability region for system (7) with added diffusion (shaded area) and without added diffusion (outlined by the dashed lines). The explicit diffusion terms added to (7a) and (7b) are similar in form to (24) with K given by (25).

The stability criterion for (26b) is found by substituting for η_j^n the Fourier term (8) to arrive at the characteristic equation

$$A - 1 = -\kappa_1(1 - e^{-ik\Delta x}) + \kappa_2(e^{ik\Delta x} - 1). \tag{28}$$

The requirement $|A|^2 \leqslant 1$ then is equivalent to the condition

$$-(\kappa_1 + \kappa_2) + \kappa_1^2 + \kappa_2^2 - 2\kappa_1\kappa_2 \cos k\Delta x \leqslant 0. \tag{29}$$

The left-hand-side is maximized for $\cos k\Delta x = -1$, so that the condition for stability becomes

$$-(\kappa_1 + \kappa_2) + (\kappa_1 + \kappa_2)^2 \leqslant 0. \tag{30}$$

This inequality is equivalent to (27) which is thus established as the stability criterion for the diffusion-augmented system.

Consideration of the case $U < 0$ yields the combined criterion (5), which determines the stability region shown by the shaded area in Fig. 2. The area represents a considerable enlargement of the region determined by (15) (outlined by the dashed line in the figure).

4. Numerical Experiments

To test the effectiveness of the approach proposed in Section 3, several numerical experiments were carried out by solving (6) with and without the supplementary diffusion terms. The dependent variables ϕ and u were assumed to be periodic of period

$2X$, and for initial conditions a "chapeau" function was used; i.e., for x in meters, we set

$$\begin{aligned}\phi(0,x) &= C_1 + C_2 x, && -2 \leqslant x \leqslant 0 \\ &= C_1 - C_2 x, && 0 < x \leqslant 2 \\ &= 0, && \text{elsewhere in } [-X, X]\end{aligned} \tag{31}$$

with

$$u(0,x) = \pm c^{-1}\,\phi(0,x). \tag{32}$$

The constants were chosen to be $X = 10$ m, $C_1 = 100\ \text{msec}^{-2}$ and $C_2 = 50\ \text{msec}^{-2}$. Selection of the plus sign in (32) suppresses the η-component of the analytic solution so that u and ϕ are functions of $x - (U + c)t$ only; similarly, choice of the minus sign implies that $\xi \equiv 0$ in which case u and ϕ are functions of $x - (U - c)t$. Condition (32) then guarantees that the initial wave form will propagate without distortion, returning to its initial position after a full period $T = 2X/(U \pm c)$. The diffusion properties of the finite-difference schemes under consideration, however, can be expected to produce numerical results that differ considerably from the analytic solution.

In the following discussion we refer to scheme (6) as scheme I ($K = 0$) and to the diffusion-augmented scheme as scheme II ($K = \frac{1}{2}c\Delta x$). The values of Δt were chosen such that.

$$\left(1 + \frac{c}{|U|}\right)(|U| + c)\frac{\Delta t}{\Delta x} = 0.8$$

for scheme I and

$$(|U| + c)\frac{\Delta t}{\Delta x} = 0.8$$

for scheme II in order to stay a safe distance within the stability region. (Such caution is really not called for here but would be advisable in practical application.)

In the set of experiments, the parametric values were selected to be

$$\begin{aligned} c = U &= 5\ \text{msec}^{-1}, & \Delta x &= 1\ \text{m}, \\ \Delta t_{\text{I}} &= 0.04\ \text{sec}, & \Delta t_{\text{II}} &= 0.08\ \text{sec}\end{aligned}$$

with the obvious subscripting convention. Figure 3a shows $\phi \times 10^{-2}$ vs x at time $t = 2$ sec for the initial condition $u(0,x) = -c^{-1}\phi(0,x)$; the solid line is the analytic solution (identical to the initial form of $\phi \times 10^{-2}$), the dotted line the scheme I solution and the dashed line the scheme II solution. For this case, $U - c = 0$ and the wave remains stationary but becomes considerably spread out by both schemes I and II with scheme II supplying slightly more diffusion. Fig. 3b shows the solutions at time $t = 2$ sec (one period) using the initial condition $u(0,x) = c^{-1}\phi(0,x)$; here the

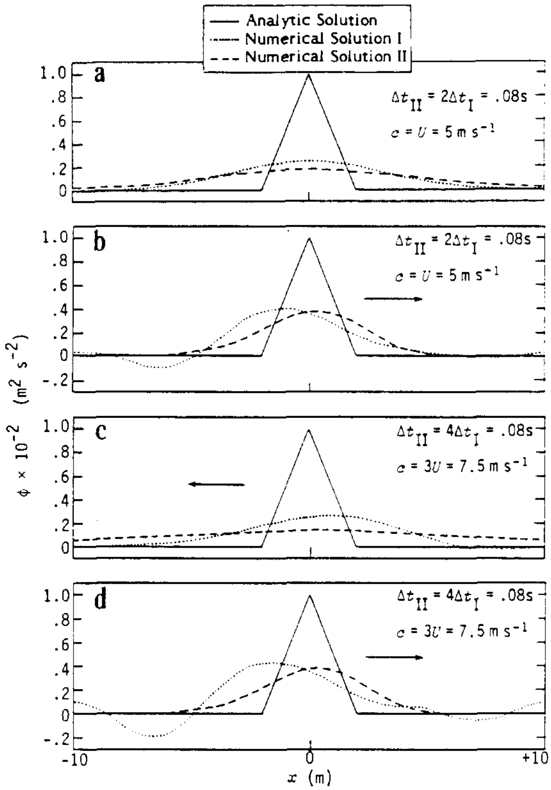

FIG. 3. Analytic solution (solid line), scheme I solution (dotted line) and scheme II solution (dashed line) for $\phi \times 10^{-2}$ as a function of x after one period for (a) $c = U = 5\ \text{msec}^{-1}$ with initial condition $u(0,x) = -c^{-1}\phi(0,x)$; (b) $c = U = 5\ \text{msec}^{-1}$ with $u(0,x) = c^{-1}\phi(0,x)$; (c) $c = 3U = 7.5\ \text{msec}^{-1}$ with $u(0,x) = -c^{-1}\phi(0,x)$; and (d) $c = 3U = 7.5\ \text{msec}^{-1}$ with $u(0,x) = c^{-1}\phi(0,x)$. Arrows indicate direction of propagation.

waves have progressed in the positive x-direction (indicated by the arrow). Scheme I introduces considerable phase lag and negative geopotential, while the scheme II solution, though somewhat damped, ends up nearly symmetrical about the origin and positive for all x.

Figures 3c and d show the results of analogous runs after one period for parametric values

$$c = 3U = 7.5\ \text{msec}^{-1}, \qquad \Delta x = 1\ \text{m},$$
$$\Delta t_I = 0.02\ \text{sec}, \qquad \Delta t_{II} = 0.08\ \text{sec}$$

Here the features, while more pronounced, are qualitatively the same as found in Figs. 3a and b.

The limited number of tests performed indicate that the more efficient scheme II is, at worst, slightly less accurate than scheme I and in some cases actually a more acceptable method. Substantial gain in efficiency can be realized by using scheme II when $c > |U|$ since we can increase the time step over that of scheme I by a factor of $1 + c/|U|$.

5. SUMMARY

In a number of meteorological modeling efforts, upstream differencing has been used to treat advection terms that appear in the primitive equations. Solutions to these equations are characterized by the presence of gravity waves. When explicit-type finite-difference schemes (such as the upstream scheme) are used to provide a numerical solution, the time step is limited not only by the magnitude of the transport velocity but also by that of the gravity-wave velocity. By neglecting the presence of gravity waves in formulating the upstream difference equations, there can result a severe restriction on the allowable size of the time step when the gravity-wave speed is large compared to the advecting speed. Violation of the criterion that places a limit on the time step gives rise to computational instability. The computational problem is brought to light through analysis of the shallow water equations, and a simple procedure is given to ease the restriction on the size of the time step.

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